

THE STONE SPACE OF THE SATURATED BOOLEAN ALGEBRAS

BY

S. NEGREPONTIS⁽¹⁾

As a consequence of results by B. Jónsson [J₁], [J₂], M. Morley and R. Vaught [MV]⁽²⁾, for every cardinal number κ such that $\kappa = \kappa^{\mathfrak{L}}$ ($= \sum \{\kappa^\mu : \mu < \kappa\}$)⁽³⁾ there is a unique (up to Boolean isomorphism) Boolean algebra \mathcal{B}_κ which is κ -homogeneous, κ -universal (in short, κ -saturated) of cardinality κ . Let S_κ be the Stone space of \mathcal{B}_κ , i.e., S_κ is a compact totally disconnected space, whose Boolean algebra of open-and-closed subsets is isomorphic to \mathcal{B}_κ . The object of the present paper is to characterize (up to homeomorphism) the space S_κ and to study the behavior of some of its subspaces. This work depends heavily on ideas that are contained in the following papers: [R₂], [P], [R₁], [HR], [K], as well as on unpublished research undertaken jointly with Professor W. W. Comfort during the summer of 1967, only a portion of which appears in [CN]. We remark that the space S_κ will have properties similar to the space $\beta\omega \setminus \omega$ of nonprincipal ultrafilters on the set ω of positive integers. Once the basic properties of S_κ are established, the arguments leading to these results are similar to the familiar arguments concerning $\beta\omega \setminus \omega$. Thus, instead of presenting the proofs in a self-contained manner, we have chosen to rely on the references mentioned above. There seem to be two main open problems concerning the spaces S_κ . These are mentioned in §2 (end remarks) and §3 (end remarks) below.

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1. A cardinal number will be an ordinal number which is an initial segment. The cardinality of a set S will be denoted by $|S|$. A Boolean algebra \mathcal{B} is κ -universal if every Boolean algebra \mathcal{C} , $|\mathcal{C}| \leq \kappa$, can be embedded into \mathcal{B} ; a Boolean algebra \mathcal{B} is

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⁽²⁾ The fact that Boolean algebras form a Jónsson class, i.e. a class satisfying conditions (I)–(VI) $_\kappa$ of [MV] involves only the simple problem of verifying the amalgamation property. The validity of this property has been noted independently by several authors: Eva Kallin (see last paragraph in [K]), Aubert Daigneault, Ph. Dwinger and F. M. Yaquib [DY], etc.

⁽³⁾ It is well known that $\kappa = \kappa^{\mathfrak{L}}$ if and only if κ is regular and such that $2^\alpha \leq \kappa$ for all cardinals $\alpha < \kappa$. Thus assuming the generalized continuum hypothesis $\kappa = \kappa^{\mathfrak{L}}$ if and only if either κ is a successor cardinal or κ is a strongly inaccessible cardinal (cf. [B₁]).

κ -homogeneous if given a Boolean algebra \mathcal{C} , $|\mathcal{C}| < \kappa$, and two embeddings $\phi, \psi: \mathcal{C} \rightarrow \mathcal{B}$, there is an automorphism h on \mathcal{B} such that $h \circ \phi = \psi$. A Boolean algebra \mathcal{B} is κ -saturated if it is κ -homogeneous and κ -universal. Unless stated otherwise, we assume from now on that $\kappa = \kappa^{\mathfrak{c}}$, and we identify the unique κ -saturated Boolean algebra of cardinality κ , \mathcal{B}_κ , with the Boolean algebra of its Stone space S_κ . For subsets \mathcal{A}, \mathcal{B} of \mathcal{B}_κ , the notation $\mathcal{A} < \mathcal{B}$ will mean that $\mathcal{A} \subsetneq \mathcal{B}$ for all $A \in \mathcal{A}, B \in \mathcal{B}$. If $\mathcal{A} = \{A\}$ we simply write $A < \mathcal{B}$ etc.

1.1. LEMMA. *For $\mathcal{A}, \mathcal{B} \subset \mathcal{B}_\kappa$, $|\mathcal{A}| + |\mathcal{B}| < \kappa$, $\mathcal{A} < \mathcal{B}$, \mathcal{A} directed upwards, \mathcal{B} directed downwards, there is $V \in \mathcal{B}_\kappa$, such that $\mathcal{A} < V < \mathcal{B}$, with the following (obvious) exceptions: if $\mathcal{B} = \emptyset$, $1 \notin \mathcal{A}$; if $\mathcal{A} = \emptyset$, $0 \notin \mathcal{B}$.*

Proof. Let S_1, S_2 be two copies of S_κ , and S the discrete union of S_1 and S_2 (i.e., S is the disjoint union of S_1, S_2 , and both S_1, S_2 are open-and-closed in S). We denote by $\mathcal{B}(S)$ the Boolean algebra of open-and-closed subsets of S , by $b(\mathcal{A} \cup \mathcal{B})$ the Boolean subalgebra of \mathcal{B}_κ generated by $\mathcal{A} \cup \mathcal{B}$. We define an embedding e of $b(\mathcal{A} \cup \mathcal{B})$ into $\mathcal{B}(S)$ by $e(A) = A_1$ for $A \in \mathcal{A}$, $e(B) = S_1 \cup B_2$ for $B \in \mathcal{B}$, and extending the definition of e in the natural way. Indeed, the extension of e can be defined in the following manner. By compactness there is $p \in \bigcap \mathcal{A} \setminus \bigcup \mathcal{B}$. Define $e: b(\mathcal{A} \cup \mathcal{B}) \rightarrow \mathcal{B}(S)$ by

$$\begin{aligned} e(x) &= x_1 && \text{if } p \notin x \\ &= S_1 \cup x_2 && \text{if } p \in x^{(4)}. \end{aligned}$$

(Here A_1 is the copy of A in S_1 , etc.) Let now \mathcal{C} be the Boolean subalgebra of $\mathcal{B}(S)$ generated by $e(\mathcal{A} \cup \mathcal{B})$ and S_1 , $\mathcal{C} = b(e(\mathcal{A} \cup \mathcal{B}) \cup \{S_1\})$. Clearly $|\mathcal{C}| < \kappa$. Let ϕ be an embedding of \mathcal{C} into \mathcal{B}_κ . The κ -homogeneity of \mathcal{B}_κ now implies the existence of an automorphism h of \mathcal{B}_κ such that $h \circ (\phi \circ e) = \text{inclusion of } b(\mathcal{A} \cup \mathcal{B}) \text{ into } \mathcal{B}_\kappa$. The definition $V = h \circ \phi(S_1)$ satisfies the conclusion of the lemma.

1.2. COROLLARY. *A maximal totally ordered subset of \mathcal{B}_κ is an η_κ -set (in the sense of Hausdorff) of cardinality κ with two endpoints.*

1.3. PROBLEM. Assuming the continuum hypothesis $\omega_1 = 2^\omega$, it can be shown that for $\kappa = \omega_1$ the property of Corollary 1.2 characterizes \mathcal{B}_{ω_1} (cf. [P]). We ask the question: If a Boolean algebra \mathcal{B} of cardinality κ satisfies the condition that every maximal totally ordered subset of \mathcal{B} is an η_κ -set with two endpoints does it follow that \mathcal{B} is isomorphic to \mathcal{B}_κ ? (We assume as usual that $\kappa = \kappa^{\mathfrak{c}}$.)

1.4. DEFINITION. For an open set U in S_κ , the type of U , $\tau(U)$ is the smallest cardinal number α , such that U can be expressed as the union of α open-and-closed subsets of S_κ (cf. [N₂]).

1.5. COROLLARY. *Let U, V be open subsets of S_κ .*

(⁴) This sentence was supplied by the referee.

- (i) If U is proper and $\tau(U) < \kappa$, then U is not dense in S_κ .
- (ii) If $\tau(U) < \kappa$, $\tau(V) < \kappa$, and $U \cap V = \emptyset$, then $\bar{U} \cap \bar{V} = \emptyset$. (\bar{U} denotes the closure of U in S_κ .)
- (iii) If $\tau(U) < \kappa$ then U is C^* -embedded in S_κ (in the sense of [GJ]).
- (iv) If $1 < \tau(U) < \kappa$, then \bar{U} is not an open subset of S_κ ; in particular S_κ is not basically disconnected (in the sense of [GJ]).

Proof. Let U be the union of $\{V_i : i \in I\}$ and V be the union of $\{W_j : j \in J\}$, where $V_i, W_j \in \mathcal{B}_\kappa$ for all i, j and $|I| < \kappa, |J| < \kappa$. For (i) let $\mathcal{A} = \{V_i : i \in I\}$, $\mathcal{B} = \{S_\kappa\}$. For (ii) let $\mathcal{A} = \{V_i : i \in I\}$, $\mathcal{B} = \{S_\kappa \setminus W_j : j \in J\}$. To prove (iv) assume that \bar{U} is open, hence $\bar{U} \in \mathcal{B}_\kappa$, and let $\mathcal{A} = \{V_i : i \in I\}$ and $\mathcal{B} = \{\bar{U}\}$. (In fact (i) is a special case of (iv).) For (iii) we use Urysohn's extension theorem, in the general form that appears as Theorem 1.17 in [GJ]. Let A, B be two completely separated subsets in U , and let f be a continuous function of U into the interval $[0, 1]$, such that $f=0$ on A and $f=1$ on B . Let $\tilde{A} = \{p \in U : f(p) < \frac{1}{2}\}$, $\tilde{B} = \{p \in U : f(p) > \frac{1}{2}\}$. It is clear that \tilde{A}, \tilde{B} are disjoint open subsets of S_κ , with $\tau(\tilde{A}) \leq \tau(U) \cdot \aleph_0 < \kappa$, $\tau(\tilde{B}) \leq \tau(U) \cdot \aleph_0 < \kappa$. By (ii), \tilde{A}, \tilde{B} have disjoint closures in S_κ , and since S_κ is compact, they are completely separated in S_κ . Thus, an application of Urysohn's extension theorem yields (iii).

Our next lemma is a direct generalization of similar lemmas that have been proved for the case $\kappa = \omega_1$ by W. Rudin [R₂], Parovičenko [P], and Keisler [K]. The proof presented below requires only slight effort over the proof of the lemma for $\kappa = \omega_1$ as it has been worked jointly with Professor W. W. Comfort (unpublished).

1.6. LEMMA. Let \mathcal{D} be a Boolean algebra satisfying Lemma 1.1 above (i.e. for $\mathcal{A}, \mathcal{B} \subset \mathcal{D}$, $|\mathcal{A}| + |\mathcal{B}| < \kappa$, $\mathcal{A} < \mathcal{B}$, there is $V \in \mathcal{D}$, such that $\mathcal{A} < V < \mathcal{B}$, with the two obvious exceptions mentioned there). Then \mathcal{D} satisfies the following (stronger) condition: for $\mathcal{A}, \mathcal{B}, \mathcal{C} \subset \mathcal{D}$, $|\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| < \kappa$, such that

- (i) $\mathcal{A} < \mathcal{B}$, \mathcal{A} directed upwards, \mathcal{B} directed downwards,
 - (ii) for all $A \in \mathcal{A}, B \in \mathcal{B}, C \in \mathcal{C}$, $C \leq A$ and $B \leq C$ are both false, there is $V \in \mathcal{D}$, such that
 - (iii) $\mathcal{A} < V < \mathcal{B}$,
 - (iv) for all $C \in \mathcal{C}$, V is not comparable to C (i.e. $V \leq C, C \leq V$ are both false).
- (Of course, we must make again the two obvious exceptions.)

Proof. We let S be the Stone space of \mathcal{D} , and we identify \mathcal{D} with the Boolean algebra $\mathcal{B}(S)$ of open-and-closed subsets of S . Thus the order \leq becomes set inclusion. Let $\lambda = |\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}|$. With possible repetitions we well-order $\mathcal{A} = \{A_\alpha\}_{\alpha < \lambda}$, $\mathcal{B} = \{B_\alpha\}_{\alpha < \lambda}$, $\mathcal{C} = \{C_\alpha\}_{\alpha < \lambda}$.

We will find nonempty open-and-closed sets F_α, G_α , $\alpha < \lambda$, such that

$$F_\alpha \subset C_\alpha \setminus \bigcup_{\beta < \lambda} A_\beta,$$

$$G_\alpha \subset \bigcap_{\beta < \lambda} B_\beta \setminus C_\alpha, \quad \text{and}$$

$$F_\alpha \cap G_\beta = \emptyset \quad \text{for all } \alpha, \beta < \lambda.$$

We proceed by transfinite induction. Let $F_1^{(1)}$ be any nonempty open-and-closed subset of $C_1 \setminus \bigcup_{\beta < \lambda} A_\beta$. Notice that this choice is possible, since by 1.5 (i) (which applies to \mathcal{D} as well as to \mathcal{B}_κ) the set $\bigcap_{\beta < \lambda} (C_1 \setminus A_\beta)$ has nonempty interior. In a similar way, let $G_1^{(1)}$ be a nonempty open-and-closed subset of $\bigcap_{\beta < \lambda} B_\beta \setminus C_1$. Notice that $F_1^{(1)} \cap G_1^{(1)} = \emptyset$. Let $\alpha < \lambda$. Suppose that for all $\gamma < \alpha$, we have defined non-empty open-and-closed sets $F_\delta^{(\gamma)}$, $G_\varepsilon^{(\gamma)}$, for all $\delta, \varepsilon \leq \gamma$, such that

$$(1) \quad F_\delta^{(\delta)} \supset F_\delta^{(\delta+1)} \supset \dots \supset F_\delta^{(\gamma)} \supset \dots, \quad G_\varepsilon^{(\varepsilon)} \supset G_\varepsilon^{(\varepsilon+1)} \supset \dots \supset G_\varepsilon^{(\gamma)} \supset \dots,$$

$$(2) \quad F_\delta^{(\gamma)} \cap G_\varepsilon^{(\gamma)} = \emptyset,$$

$$(3) \quad F_\gamma^{(\gamma)} \subset C_\gamma \setminus \bigcup_{\beta < \lambda} A_\beta, \quad G_\gamma^{(\gamma)} \subset \bigcap_{\beta < \lambda} B_\beta \setminus C_\gamma.$$

Let $E_\gamma^{(\alpha)}$ be a nonempty open-and-closed subset of $\bigcap_{\delta < \alpha} F_\gamma^{(\delta)}$, and let $H_\gamma^{(\alpha)}$ be a non-empty open-and-closed subset of $\bigcap_{\delta < \alpha} G_\gamma^{(\delta)}$; the choices are possible by 1.5 (i). Notice that if $\alpha = \beta + 1$, we may simply take $E_\gamma^{(\alpha)} = F_\gamma^{(\beta)}$, $H_\gamma^{(\alpha)} = G_\gamma^{(\beta)}$. At this point we employ a "subsidiary" transfinite induction (the details will be omitted) to define $F_\alpha^{(\alpha)}$ to be any nonempty open-and-closed subset of $C_\alpha \setminus \bigcup_{\beta < \lambda} A_\beta$, such that $H_\gamma^{(\alpha)} \setminus F_\alpha^{(\alpha)} \neq \emptyset$ for all $\gamma < \alpha$, and $G_\alpha^{(\alpha)}$ to be a nonempty open-and-closed subset of $\bigcap_{\beta < \lambda} B_\beta \setminus C_\alpha$, such that $E_\gamma^{(\alpha)} \setminus G_\alpha^{(\alpha)} \neq \emptyset$ for all $\gamma < \alpha$. We then set $G_\gamma^{(\alpha)} = H_\gamma^{(\alpha)} \setminus F_\alpha^{(\alpha)}$, $F_\gamma^{(\alpha)} = E_\gamma^{(\alpha)} \setminus G_\alpha^{(\alpha)}$ for all $\gamma < \alpha$. It can be seen that conditions (1), (2), (3) above are satisfied for α . Finally we let F_α , G_α be any nonempty open-and-closed subsets of $\bigcap_{\delta < \lambda} F_\alpha^{(\delta)}$, $\bigcap_{\delta < \lambda} G_\alpha^{(\delta)}$, respectively, using again 1.5(i).

Let now $\tilde{\mathcal{A}} = \{A_{\alpha_1} \cup G_{\alpha_1} \cup \dots \cup A_{\alpha_n} \cup G_{\alpha_n} : \alpha_1, \dots, \alpha_n < \lambda, n < \omega\}$,

$$\tilde{\mathcal{B}} = \{(B_{\alpha_1} \setminus F_{\alpha_1}) \cap \dots \cap (B_{\alpha_n} \setminus F_{\alpha_n}) : \alpha_1, \dots, \alpha_n < \lambda, n < \omega\};$$

it is clear that $\tilde{\mathcal{A}} < \tilde{\mathcal{B}}$, $\tilde{\mathcal{A}}$ directed upwards, $\tilde{\mathcal{B}}$ directed downwards, and thus there is a nonempty open-and-closed set V , such that $\tilde{\mathcal{A}} < V < \tilde{\mathcal{B}}$; the set V satisfies the conditions of the lemma.

The following result provides a topological characterization of the Stone space S_κ .

1.7. THEOREM. *The space S_κ is a compact, totally disconnected space, having κ open-and-closed sets, and with no isolated points, satisfying the following condition:*

Every proper open set of type less than κ is not dense and it is C^ -embedded.*

Moreover, any two spaces satisfying these conditions are homeomorphic.

Proof. That S_κ satisfies the above conditions follows from 1.1, 1.5. The uniqueness follows by a familiar transfinite induction (cf. [R₂], [P], [HR], [K]) using Lemma 1.6, and will be omitted.

A partial order \geq on a set P will be assumed to satisfy in addition to the axioms of reflexivity and transitivity, the antisymmetric axiom: $x \geq y$, $y \geq x$ implies $x = y$. It was shown in [J₁] that the system of partially ordered sets is a Jónsson class (i.e. it satisfies (I)–(VI) $_\kappa$ in [MV]) and, hence, admits saturated structures.

1.8. THEOREM. *As a partially ordered set \mathcal{B}_κ is κ -universal.*

The proof follows from 1.6, the fact that every partially ordered set can be embedded in a directed partially ordered set, and applications of well-known arguments and, thus, will be omitted.

We remark, however, that as a partially ordered set, \mathcal{B}_κ is not κ -homogeneous, since it is clear that the κ -saturated partially ordered set cannot have a first, or last element. Even if we delete the endpoints of \mathcal{B}_κ , the resulting set fails to be the κ -saturated partially ordered set⁽⁵⁾. Indeed, it is easy to verify that a characteristic property of the κ -saturated partially ordered set \mathcal{P}_κ is the following: given \mathcal{A} , \mathcal{B} , $\mathcal{C} \subset \mathcal{P}_\kappa$, such that $|\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| < \kappa$, $\mathcal{A} < \mathcal{B}$, and the relations $a \leq c$, $c \leq b$ are false for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, $c \in \mathcal{C}$, then there is $v \in \mathcal{P}_\kappa$, such that $\mathcal{A} < v < \mathcal{B}$, and such that the relations $v \leq c$, $c \leq v$ are false for all $c \in \mathcal{C}$. It is to be noted that \mathcal{B}_κ satisfies the above property, according to Lemma 1.6 above, only for sets \mathcal{A} , \mathcal{B} , \mathcal{C} as above, that in addition have \mathcal{A} directed upwards and \mathcal{B} directed downwards.

2. By Stone's duality between Boolean algebras and totally disconnected compact spaces (in a categorical sense) the following statements are true for S_κ :

(a) Every compact totally disconnected space having at most κ open-and-closed sets is the continuous epimorphic image of S_κ .

(b) For any compact totally disconnected space K , with less than κ open-and-closed sets, and any two continuous epimorphisms f, g of S_κ onto K , there is an autohomeomorphism h of S_κ , such that $g \circ h = f$.

We improve these statements by omitting the restriction that K is totally disconnected, with the aid of the following simple lemmas. For a topological space K , the weight $w(K)$ of K is the smallest cardinal number α , such that K has a base for its topology of cardinality α .

2.1. LEMMA. *Let f be an epimorphism of the compact totally disconnected space S onto the compact space K . There is a compact totally disconnected space E , with $w(K)$ open-and-closed subsets, and epimorphisms $g: S \rightarrow E$, $h: E \rightarrow K$, such that $h \circ g = f$.*

Proof. We may assume that $w(K)$ is infinite. Let \mathcal{A} be a base for the topology, consisting of cozero-sets (cf. [GJ]) and having cardinality $w(K)$. For $A \in \mathcal{A}$, it is easy to find a sequence of open-and-closed subsets of S , $V_{i,A}$, $i \in \omega$, such that $f^{-1}(A) = \bigcup \{V_{i,A} : i \in \omega\}$. Let \mathcal{B} be the Boolean subalgebra of open-and-closed subsets of S , generated by all sets of the form $V_{i,A}$ for $A \in \mathcal{A}$, $i \in \omega$, and let E be the Stone space of \mathcal{B} . Since \mathcal{B} is a subalgebra of the Boolean algebra of S , there is an epimorphism $g: S \rightarrow E$ induced by the inclusion of algebras. By the definition of E ,

⁽⁵⁾ The fact that \mathcal{B}_κ is not κ -homogeneous as a partially ordered set was mentioned to the author by the referee. The fact that \mathcal{B}_κ , even with its endpoints removed, is not κ -homogeneous as a partially ordered set was remarked to the author by Professor M. Morley.

it is clear that for every $e \in E$, the function f is constant on $g^{-1}(e)$. Let $h: E \rightarrow K$ be given by $h(e) = f(g^{-1}(e))$. Since g is a quotient mapping, h is continuous. It is clear that $|\mathcal{B}| = |\mathcal{A}| = w(K)$.

The second lemma is related to the amalgamation property for Boolean algebras (cf. [DY]).

2.2. LEMMA. *Given compact spaces E_1, E_2, K and continuous epimorphisms $h_1: E_1 \rightarrow K, h_2: E_2 \rightarrow K$, there is a compact space E , such that $w(E) = w(E_1) + w(E_2)$, and continuous epimorphisms $k_1: E \rightarrow E_1, k_2: E \rightarrow E_2$, such that $h_1 \circ k_1 = h_2 \circ k_2$. Moreover, if E_1, E_2 are totally disconnected, so is E .*

Proof. Let $E = \{(x, y) \in E_1 \times E_2 : h_1(x) = h_2(y)\}$. It is clear that E is compact, and that it is totally disconnected if E_1, E_2 are totally disconnected. Let k_1, k_2 be the restrictions to E of the usual projections of $E_1 \times E_2$ onto E_1, E_2 respectively. Since h_1, h_2 are epimorphic, it is clear that k_1, k_2 are epimorphic. Thus $w(E) = w(E_1) + w(E_2)$, and $h_1 \circ k_1 = h_2 \circ k_2$.

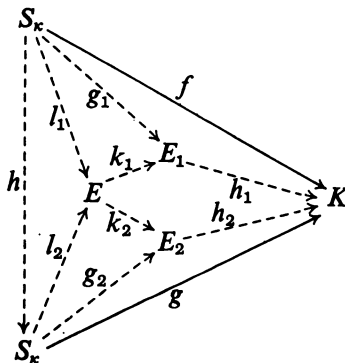
The next lemma is a special case of Corollary 2.4 (d) in [MV] which we state here for reference.

2.3. LEMMA. *Given compact totally disconnected spaces E, E_1 , each with less than κ open-and-closed sets, and continuous epimorphisms $g: S_\kappa \rightarrow E_1, k: E \rightarrow E_1$, there is a continuous epimorphism $l: S_\kappa \rightarrow E$, such that $k \circ l = g$.*

We now prove the statement corresponding to (b) above, and which may be called (if consistency is required) the κ -co-homogeneity of S_κ for compact spaces.

2.4. THEOREM. *Whenever K is a compact space of weight less than κ , and f, g are continuous epimorphisms of S_κ onto K , there is an autohomeomorphism h on S_κ such that $g \circ h = f$.*

Proof. We will refer to the following diagram:



By Lemma 2.1, there are compact totally disconnected spaces E_1, E_2 , with $w(K)$ open-and-closed sets, and epimorphisms $g_1: S_\kappa \rightarrow E_1, h_1: E_1 \rightarrow K, g_2: S_\kappa \rightarrow E_2, h_2: E_2 \rightarrow K$ such that $h_1 \circ g_1 = f, h_2 \circ g_2 = g$. By 2.2, there is a compact totally dis-

connected space E , with $w(K)$ open-and-closed sets, and epimorphisms $k_1: E \rightarrow E_1$, $k_2: E \rightarrow E_2$, such that $h_1 \circ k_1 = h_2 \circ k_2$. By 2.3, there are epimorphisms $l_1, l_2: S_\kappa \rightarrow E$ such that $k_1 \circ l_1 = g_1$, $k_2 \circ l_2 = g_2$. Finally, by property (b) above, since $w(K) < \kappa$, there is an autohomeomorphism h of S_κ , such that $l_2 \circ h = l_1$. Thus $g \circ h = h_2 \circ g_2 \circ h = (h_2 \circ k_2) \circ (l_2 \circ h) = h_1 \circ k_1 \circ l_1 = h_1 \circ g_1 = f$.

2.5. LEMMA. *For any compact space K , there is a compact totally disconnected space S , with $w(K)$ open-and-closed sets, and a continuous epimorphism h of S onto K .*

Proof. Let ϕ be a continuous epimorphism of the Cantor set C onto the unit interval I , and Φ the induced continuous epimorphism of $C^{w(K)}$ onto $I^{w(K)}$. It is clear that K may be considered as a closed subset of $I^{w(K)}$. Let S be the preimage, under Φ , of K , and $h = \Phi|_S$.

The resulting statement is the analogue of (a) above and may be called the κ -co-universality of S_κ for compact spaces.

2.6. THEOREM. *Every compact space of weight at most κ is the continuous epimorphic image of S_κ .*

For the proof we use 2.5 and (a).

For completeness we state the following consequence of [MV], [DY] and 2.4, 2.6 above, which provides a second description of S_κ .

2.7. THEOREM. *S_κ is the unique (up to homeomorphism) compact space of weight κ , which is κ -co-homogeneous and κ -co-universal.*

By Keisler [K], S_{ω_1} can be identified with $\beta\omega \setminus \omega$ (under the continuum hypothesis $\omega_1 = 2^\omega$). As a particular case of 2.7, $\beta\omega \setminus \omega$ can be characterized uniquely up to homeomorphism by the following properties (assuming the continuum hypothesis):

- (a) $\beta\omega \setminus \omega$ is a compact space of weight ω_1 ,
- (b) every compact metric space is the continuous epimorphic image of $\beta\omega \setminus \omega$, and

(c) $\beta\omega \setminus \omega$ is projective in the category of compact metric spaces and continuous epimorphisms in the following sense: given compact metric spaces X , Y , and continuous epimorphisms $g: \beta\omega \setminus \omega \rightarrow Y$, $f: X \rightarrow Y$, there is a continuous epimorphism $h: \beta\omega \setminus \omega \rightarrow X$, such that $f \circ h = g$.

Esenin-Vol'pin [E] assuming the generalized continuum hypothesis has proved the existence, for every cardinal κ (not necessarily $\kappa = \kappa^\mathfrak{c}$), of a compact space of weight κ , which is, in the above sense, κ -co-universal for compact spaces. Assume the generalized continuum hypothesis, and let κ be any cardinal number. If κ is regular (in particular if $\kappa = \alpha^+$ for some α) it follows from Theorem 2.7 that, in fact, S_κ is up to homeomorphism the only κ -co-homogeneous and κ -co-universal compact space of weight κ .

If κ is singular the dual notion of κ -special of Morley and Vaught [MV] may be added to κ -co-universality to produce a unique compact space.

As it is remarked in [MV] the results of Sierpiński [S₁] concerning universal metric spaces can be improved in a similar way. Since compact spaces are related in a dual way both to Boolean algebras and to Banach spaces, and since, according to [K] (and implicitly according to [R₂], [P], [HR]) the space $\beta\omega \setminus \omega$ is the ω_1 -co-universal, ω_1 -co-homogeneous compact space of cardinality ω_1 , it is natural to ask whether the Banach space $C(\beta\omega \setminus \omega)$ of continuous bounded real-valued functions on $\beta\omega \setminus \omega$ is the ω_1 -saturated metric space of cardinality ω_1 . At least the property of ω_1 -homogeneity fails according to an obvious modification of a simple idea of Sierpiński [S₁]. It is, in fact, not known to the author whether the κ -saturated metric spaces of cardinality κ have the structure of a Banach space. But the idea of Sierpiński is sufficient to show that they cannot have the form $C(S)$ for a compact space S .

The following problem is open: let $\kappa = \alpha^+ = 2^\alpha$. Is it possible to represent S_κ as a space of the form $\beta X_\kappa \setminus X_\kappa$, where X_κ is a locally compact, realcompact (or even paracompact) space, which can be expressed as the union of (at most) α compact spaces? For $\kappa = \omega_1$, we simply take $X_\kappa = \omega$. Unfortunately, as it has been remarked by Keisler [K], and as it also follows from [FG], [N₂], the choice $X_\kappa = \alpha$ does not work. In fact, even the space of uniform ultrafilters on α is not suitable. A possible choice is to take X_κ as an open subset of S_κ of type α , e.g. the pairwise disjoint union of α open-and-closed nonempty subsets of S_κ . As it may be seen from [CN] this is suitable for $\kappa = \omega_1$.

3. Open subspace of S_κ . Let U be an open subset of S_κ . If $\tau(U) < \kappa$, by 1.5 (iv), U is C^* -embedded in S_κ . Our initial result below will establish a much stronger property for such open sets U , i.e. that for any compact space K , and any continuous mapping $f: U \rightarrow K$, there is a continuous extension $g: S_\kappa \rightarrow K$ of f . As a consequence \bar{U} will be a retract of S_κ . In contrast, if $\tau(U) = \kappa$, there is a two-valued continuous function on U which cannot be continuously extended to S_κ (cf. [FG]).

3.1. THEOREM. *Let U be an open subspace of S_κ , $\tau(U) < \kappa$, and let f be a continuous function of U into any compact space K . There is a continuous extension g of f , $g: S_\kappa \rightarrow K$.*

Proof. Since U is C^* -embedded in S_κ , $\bar{U} = \beta U$, and \bar{U} has at most κ open-and-closed sets. It is clear, therefore, that we may assume that the weight of K is at most κ and that f maps U onto a dense subspace of K . We first prove the result for the case that K is totally disconnected. The Boolean algebra \mathcal{C} of open-and-closed sets of K has cardinality at most κ . Let $\mathcal{B}(U)$ be the Boolean algebra of open-and-closed subsets of U ; since $\bar{U} = \beta U$, we may identify $\mathcal{B}(U)$ with $\mathcal{B}(\bar{U})$, and the restriction mapping $R: \mathcal{B}_\kappa \rightarrow \mathcal{B}(U)$ is an epimorphism. Further, let $f^\#: \mathcal{C} \rightarrow \mathcal{B}(U)$ be the monomorphism given by $f^\#(C) = f^{-1}(C)$. We wish to define a monomorphism $e: \mathcal{C} \rightarrow \mathcal{B}_\kappa$, such that $R \circ e = f^\#$. Let $\{C_\alpha\}_{\alpha < \kappa}$ be a well-ordering of \mathcal{C} , with repetitions if necessary, $C_0 = \emptyset$, \mathcal{C}_α = the Boolean subalgebra of \mathcal{C} generated by $\{C_\gamma\}_{\gamma \leq \alpha}$, \mathcal{C}_α = the Boolean subalgebra of \mathcal{C} generated by $\{C_\gamma\}_{\gamma < \alpha}$. Let $e_0(C_0) = \emptyset$,

$e_0(K) = S_\kappa$. Let $\alpha < \kappa$, and assume that for all $\gamma < \alpha$, we have defined embeddings $e_\gamma: \mathcal{C}_\gamma \rightarrow \mathcal{B}_\kappa$ such that

- (i) $e_\delta \subset e_\gamma$ for $\delta < \gamma < \alpha$,
- (ii) $e_\gamma(A) \cap U = f^{-1}(A)$ for all $A \in \mathcal{C}_\gamma$,
- (iii) $e_\gamma(A) = f^{-1}(A)$ for all $A \in \mathcal{C}_\gamma$ for which $f^{-1}(A) \in \mathcal{B}_\kappa$.

Notice that $\tilde{\mathcal{C}}_\alpha = \bigcup_{\gamma < \alpha} \mathcal{C}_\gamma$, and let $\tilde{e}_\alpha = \bigcup_{\gamma < \alpha} e_\gamma: \tilde{\mathcal{C}}_\alpha \rightarrow \mathcal{B}_\kappa$. Clearly \tilde{e}_α is an embedding.

Case 1. If $C_\alpha \in \tilde{\mathcal{C}}_\alpha$, $\mathcal{C}_\alpha = \tilde{\mathcal{C}}_\alpha$, we set $e_\alpha = \tilde{e}_\alpha$.

Suppose now that $C_\alpha \notin \tilde{\mathcal{C}}_\alpha$. Then \mathcal{C}_α is the Boolean subalgebra of \mathcal{C} generated by $\tilde{\mathcal{C}}_\alpha$ and C_α , i.e. $\mathcal{C}_\alpha = \{(A \cap C_\alpha) \cup (B \setminus C_\alpha) : A, B \in \tilde{\mathcal{C}}_\alpha\}$.

Case 2. There is $\gamma < \alpha$, such that

$$C_\gamma \subset C_\alpha \quad \text{and} \quad f^{-1}(C_\alpha) \setminus f^{-1}(C_\gamma) \in \mathcal{B}_\kappa$$

(i.e. $f^{-1}(C_\alpha) \setminus f^{-1}(C_\gamma)$ is compact). Let $e_\alpha(C_\alpha) = (f^{-1}(C_\alpha) \setminus f^{-1}(C_\gamma)) \cup e_\gamma(C_\gamma)$, $e_\alpha((A \cap C_\alpha) \cup (B \setminus C_\alpha)) = [\tilde{e}_\alpha(A) \cap e_\alpha(C_\alpha)] \cup [\tilde{e}_\alpha(B) \setminus e_\alpha(C_\alpha)]$, for $A, B \in \tilde{\mathcal{C}}_\alpha$.

Case 3. There is $\gamma < \alpha$, such that

$$C_\alpha \subset C_\gamma \quad \text{and} \quad f^{-1}(C_\gamma) \setminus f^{-1}(C_\alpha) \in \mathcal{B}_\kappa$$

(i.e. $f^{-1}(C_\gamma) \setminus f^{-1}(C_\alpha)$ is compact). Let $e_\alpha(C_\alpha) = e_\gamma(C_\gamma) \setminus f^{-1}(C_\alpha)$, etc. The verification that in Cases 2 and 3 e_α is a well-defined embedding, satisfying (i), (ii), (iii) is not difficult.

Case 4. Suppose that the above three cases fail. Notice that $f^{-1}(C_\alpha)$ is an open-and-closed subset of U and hence $\tau(f^{-1}(C_\alpha)) < \kappa$, $\tau(U \setminus f^{-1}(C_\alpha)) < \kappa$. Let $\{V_i : i \in I\}$, $\{W_j : j \in J\}$ be two families of open-and-closed subsets of S_κ , directed upwards, and such that $|I| + |J| < \kappa$, $\bigcup_{i \in I} V_i = f^{-1}(C_\alpha)$, $\bigcup_{j \in J} W_j = U \setminus f^{-1}(C_\alpha)$. Set

$$\mathcal{A} = \{e_\gamma(C_\gamma) \cup V_i : C_\gamma \subset C_\alpha, \gamma < \alpha, i \in I\},$$

$$\mathcal{B} = \{e_\gamma(C_\gamma) \setminus W_j : C_\alpha \subset C_\gamma, \gamma < \alpha, j \in J\},$$

and

$$\mathcal{C} = \{e_\gamma(C_\gamma) : (C_\gamma \setminus C_\alpha) \cup (C_\alpha \setminus C_\gamma) \neq \emptyset, \gamma < \alpha\}.$$

Notice that $|\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| < \kappa$, that \mathcal{A} is directed upwards, \mathcal{B} is directed downwards, and $\mathcal{A} < \mathcal{B}$, and that for $A \in \mathcal{A}$, $B \in \mathcal{B}$, $C \in \mathcal{C}$, $C \not\leq A$, $B \not\leq C$. Hence, by Lemma 1.6, there is an open-and-closed set $e_\alpha(C_\alpha)$ such that $\mathcal{A} < e_\alpha(C_\alpha) < \mathcal{B}$, and $e_\alpha(C_\alpha)$ is comparable with no element of \mathcal{C} . Notice that $V_i \subset e_\alpha(C_\alpha) \subset S_\kappa \setminus W_j$ for all $i \in I, j \in J$, and hence $e_\alpha(C_\alpha) \cap U = f^{-1}(C_\alpha)$. Define now $e_\alpha: \mathcal{C}_\alpha \rightarrow \mathcal{B}_\kappa$ by

$$e_\alpha((A \cap C_\alpha) \cup (B \setminus C_\alpha)) = [\tilde{e}_\alpha(A) \cap e_\alpha(C_\alpha)] \cup [\tilde{e}_\alpha(B) \setminus e_\alpha(C_\alpha)] \quad \text{for } A, B \in \tilde{\mathcal{C}}_\alpha.$$

The verification that e_α is an embedding satisfying (i), (ii), (iii) requires a lengthy, but routine argument which will be omitted. Finally, we set $e = \bigcup_{\alpha < \kappa} e_\alpha: \mathcal{C} \rightarrow \mathcal{B}_\kappa$, an embedding of \mathcal{C} into \mathcal{B}_κ such that $R \circ e = f^\#$. Let g be the dual (to e) continuous epimorphism of S_κ onto K ; by Stone's duality for Boolean algebras g is an extension of f .

Consider now the general case with K any compact space. By 2.1 there is a totally disconnected compact space D , and epimorphisms $g: \bar{U} \rightarrow D$, $h: D \rightarrow K$, such that $h \circ g = f^\beta$ = the Stone extension of f to $\beta U = \bar{U}$. By our argument above, there is a continuous extension \bar{g} of g over S_κ . The mapping $\bar{f} = h \circ \bar{g}$ provides the required extension of f over S_κ .

3.2. COROLLARY. *Let F be a closed subset of S_κ containing densely an open subspace U , $\tau(U) < \kappa$. Then there is a retraction of S_κ onto F .*

Proof. Apply 3.1 to the inclusion of U into F .

For example, the closure of a cozero-set in $\beta\omega \setminus \omega$ (or, of S_κ) is a retract of $\beta\omega \setminus \omega$ (or, of S_κ).

The following theorem is proved in a similar way as Theorem 3.1, by a “back-and-forth” induction. For $\kappa = \omega_1$ cf. Theorem 9 in [R₁], and the remarks following Theorem 2.1 in [CN]. The details will be omitted.

3.3. THEOREM. *Let U, V be proper open subsets of S_κ , $\tau(U) < \kappa$, $\tau(V) < \kappa$, and h a homeomorphism of U onto V . There is an autohomeomorphism g of S_κ , such that $g|U = h$.*

3.4. COROLLARY. *There is a space Ω_κ expressible in the form $\Omega_\kappa = \bigcup_{\alpha < \kappa} (S_\kappa)_\alpha$, where each $(S_\kappa)_\alpha$ is a homeomorph of S_κ , open-and-closed in Ω_κ , and such that $(S_\kappa)_\alpha \not\supseteq \bigcup_{\gamma < \alpha} (S_\kappa)_\gamma$. Furthermore, up to homeomorphism Ω_κ is the only such space.*

The proof of this statement is an easy application of 3.3. In order to examine some copies of Ω_κ in S_κ we introduce the following definitions. A space is of “ κ Baire category” if the intersection of at most κ open dense subsets is dense in the space. A point p in a space S is a P_κ -point if the intersection of less than κ neighborhoods of p is a neighborhood of p . The usual P -point, as defined in [GJ], is in the above notation a P_{ω_1} -point. As it has been proved by W. Rudin [R₂], $\beta\omega \setminus \omega$ has a dense set of 2^c P_{ω_1} -points. As it has been remarked by Parovičenko [P], $\beta\omega \setminus \omega$ is of ω_1 Baire category. For $\kappa = \omega_1$, the fact that $\beta\omega \setminus \omega$ is not point-homogeneous has been recently proved by Z. Frolík [F], without any use of the continuum hypothesis. His proof does not establish however the existence of P -points in $\beta\omega \setminus \omega$. It has been announced by D. D. Booth [B₂] that the existence of P -points in $\beta\omega \setminus \omega$ is consistent with certain denials of the continuum hypothesis.

3.5. THEOREM. *Let $\kappa > \omega$. S_κ is of κ Baire category, and it contains a dense set of 2^κ P_κ -points. In particular, S_κ is not point-homogeneous.*

The proof of the first statement can be easily given, using 1.5. For the existence of P_κ -points, notice that a point $p \in S_\kappa$ is a P_κ -point if and only if p does not belong to the boundary of an open set U , $\tau(U) < \kappa$. Thus the set P_κ of P_κ -points is equal to the intersection of all sets of the form $S_\kappa \setminus (\bar{U} \setminus U)$, where U is an open set, $\tau(U) < \kappa$. The number of such sets U is $\kappa^\kappa = \kappa$ and thus P_κ is a dense subset of S_κ . It can

easily be seen that $|P_\kappa| = 2^\kappa$. Since S_κ is compact, $P_\kappa \neq S_\kappa$, and since under a homeomorphism a P_κ -point must be mapped to a P_κ -point, S_κ is not point-homogeneous.

3.6. COROLLARY. *Let U be an open set of S_κ , $1 < \tau(U) < \kappa$, and p a P_κ -point. Then the spaces $S_\kappa - \bar{U}$, $S_\kappa - \{p\}$ are each homeomorphic to Ω_κ (defined in 3.4).*

For the proof use 1.5 and 3.4. See also 2.2 in [CN], for the case $\kappa = \omega_1$.

3.7. THEOREM. *If U is an open subset of S_κ , $\tau(U) = \kappa$, then there is a two-valued continuous function on U which cannot be extended to S_κ .*

The proof follows in the same way as in [FG]; see also [N₂] for a generalization of [FG] in a different direction.

Let Λ_κ be the set 2^κ of all $\{0, 1\}$ -sequences indexed by κ , ordered lexicographically and endowed with the order topology. Since Λ_κ is Dedekind complete, with a first and last element, it is compact in its order topology. For any space S , we let $P_\kappa(S)$ denote the set of P_κ -points in S (thus $P_\kappa = P_\kappa(S_\kappa)$) and we let $\pi_\kappa(S)$ denote the set S , with a base for the topology consisting of the family of all intersections of less than κ open sets. Every element of $\pi_\kappa(S)$ is a P_κ -point. We state here the extension of the results of Parovičenko [P], omitting the similar proofs (cf. [CN]).

3.8. THEOREM. *The following spaces are all homeomorphic: $\pi_\kappa(\Lambda_\kappa)$, $P_\kappa(\Lambda_\kappa)$, P_κ , $\pi_\kappa(S_\kappa)$. In particular, S_κ contains densely an η_κ -set of cardinality κ .*

We conclude with a problem. The space S_κ has not been constructed in an explicit way, and its topological properties were consequences of the fact that \mathcal{B}_κ is κ -saturated. For $\kappa = \omega_1$, the natural homeomorph of S_κ is $\beta\omega \setminus \omega$. (As it has been noted above, the explicit statement of this fact is due to Keisler [K].) Theorem 3.8 suggests the construction of S_κ as the compactification of an η_κ -set of cardinality κ . We have not been able to carry out such a construction. We note that for $\kappa > \omega$ the two "natural" compactifications of an η_κ -set of cardinality κ , namely the Stone-Čech compactification and the ordered compactification (which can be identified with the space Λ_κ minus its set of upper points in the terminology of Gillman and Jerison [GJ]) are not homeomorphic to S_κ . The Stone-Čech compactification has cardinality 2^{2^κ} , while S_κ has cardinality 2^κ . The ordered compactification cannot be even an F -space in the terminology of [GJ], since otherwise it would have to be a P -space (see Problem 5O in [GJ]), and that is not compatible with compactness.

4. We conclude with some remarks on the closed extremally disconnected subspaces of S_κ . A space is extremally disconnected if every open set has an open closure. The injective objects in the category of Boolean algebras are precisely the complete Boolean algebras (Sikorski [S₂]). A more general result (Gleason [G]) identifies the projective objects in the category of compact spaces with the extremally disconnected spaces. Incidentally, we remark that Sikorski's theorem, Stone's duality for Boolean algebras, together with Lemmas 2.1, 2.2 actually imply Gleason's theorem.

4.1. THEOREM. (a) *If E is an extremally disconnected compact space, with at most κ open-and-closed sets, then E can be embedded as a retract in S_κ .*

(b) *If E is homeomorphic to a retract of S_κ , with less than κ open-and-closed sets, then E is extremally disconnected.*

Proof. (a) There is a continuous epimorphism $f: S_\kappa \rightarrow E$. Let E' be a closed subset of S_κ , such that $f|_{E'}$ is irreducible, i.e. $f(E') = E$, and $f(F) \neq E$ for every proper closed subset of E' . By Gleason's Lemma [G], $f|_{E'}$ is a homeomorphism. It is easy now to see that E' is a retract of S (cf. [N₁]).

(b) Using 2.1, 2.2, 2.3 we can prove that S_κ is projective in the category of all compact spaces of weight less than κ . A retract E of a projective object is projective. If in addition E has less than κ open-and-closed sets, then it is easy to prove that E is projective in the category of all compact spaces (and continuous maps), and thus by Gleason's Theorem, E is extremally disconnected.

Note that 4.1 does not claim that if E is a closed subset of S_κ , E extremally disconnected (even with less than κ open-and-closed sets), then E is a retract of S_κ . This must be left as a question.

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McGILL UNIVERSITY,
MONTRÉAL, QUÉBEC